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LETTER TO THE EDITOR

Scaling of the first-passage time of biased diffusion on hierarchical comb structures

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Abstract. Biased diffusion on hierarchical comb structures is studied within an exact renormalisation group scheme. The scaling exponents of the moments of the first-passage time for random walks are obtained. It is found that the scaling properties of the diffusion depend only on the direction of bias. In a particular case, the presence of bias may give rise to a new multifractality.

The scaling properties of diffusion on disordered structures and fractal structures have recently attracted much attention [1–6]. In a recent work, Havlin *et al* [3] studied the moments of the first-passage time (FPT) for random walks on a family of hierarchical comb structures (see figure 1) using the exact enumeration method. Here FPT means the time needed for a random walker to pass the Nth-order hierarchical comb. It was shown that the scalings of the different moments of FPT as a function of the size of the system L are characterised by a set of exponents and not by a single exponent, i.e. the diffusion is of a multifractal nature. Later, this model was exactly solved by Kahang *et al* within a renormalisation group treatment [4]. The scaling exponent τ_q for the qth moment of FPT $\langle t^q \rangle \sim L^{\tau_q}$ was found to be

$$\tau_q = \begin{cases} \ln 2R^{2q-1}/\ln 2 & R>2 \\ 2q & R\leqslant 2\ddagger \end{cases} \qquad (1)$$

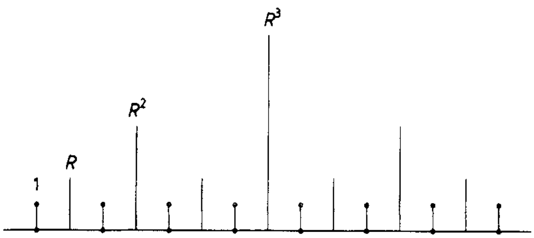

Figure 1. Hierarchical comb structure. The random walker makes unit steps on the backbone and on the teeth of the structure. The powers R^n represent the teeth length in the structure. The third-order hierarchical comb structure is shown here and the sites marked by circles are decimated in the renormalisation.

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‡ Herein and hereinafter τ_q for $R=2$ only means $\langle t^q \rangle_{N+1} \sim 2^{\tau_1} \langle t^q \rangle_N$ for very large N. In this sense, the moments of FPT for random walks on the $R=2$ hierarchical comb are characterised by a single exponent as $\tau_q = q\tau_1$, thus there is no multifractility of the transit time moments.

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where the hierarchical parameter R characterises the tooth length in the hierarchical comb (see figure 1). In addition to the multifractal behaviour of diffusion, as $\tau_q \neq q\tau_1$, a dynamical phase transition from anomalous to ordinary diffusion was found at $R = R_c = 2$.

In this letter, we study diffusion on the hierarchical comb structures in the presence of the external force which may bias the diffusion in various directions. The scaling exponents of the moments of FPT are obtained by an exact renormalisation method. It is found that the mean FPT may grow with the size of the system as a power law with different exponents or even exponentially in the presence of bias. The scaling properties of diffusion depend only on the direction of the bias. Further, the presence of bias could not only eliminate multifractality of the diffusion, but also give rise to the multifractal behaviour as $\tau_q \neq q\tau_1$ in the particular case of $R = 2$.

As usual, let $P_m(n)$ be the probability of the random walker being at site m on the nth step and $P_m(z)$ the corresponding generating function $P_m(z) = \Sigma_{n=0}^{\infty} P_m(n) z^n$. For the generally biased diffusion, we need 13 hopping rates, defined in figure 2, as a minimal set to close a direct renormalisation from an Nth-order to an $(N-1)$th-order hierarchical comb. With these hopping rates, the master equations for the generating function for the zeroth-order hierarchical comb (see figure 2(*b*)) are

$$\begin{aligned} P_1(z) &= aP_2(z) + \Delta \qquad & P_2(z) &= cP_1(z) + dP_4(z) \\ P_3(z) &= gP_2(z) \qquad & P_4(z) &= eP_2(z) \end{aligned} \tag{2}$$

with the solution for the first-passage probability

$$P_3(z) = \frac{cg\Delta}{1 - ac - de} \tag{3}$$

where the parameter Δ means that the particle is supposed to start its diffusion from site 1.

To be specific, we discuss the renormalisation of an $R = 3$ hierarchical comb. The renormalisation for any other finite R is similar. For an Nth-order comb, by decimating the Nth-order sites such as marked by circles in figure 1 (for $N = 3$) and relabelling

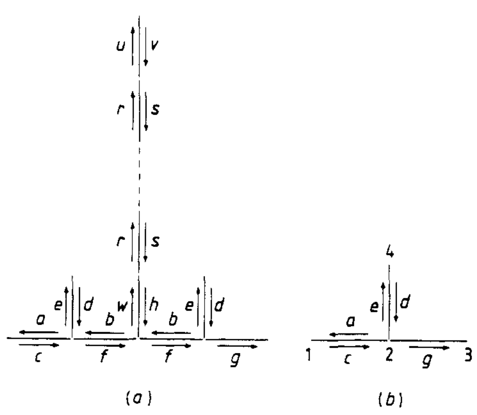

Figure 2. (*a*) A first-order hierarchical comb structure showing the set of 13 hopping rates needed to close the renormalisation of the structure. (*b*) The zero-order structure showing that only five hopping rates are needed for the solution to the first-passage probability $P_3(z)$.

the site number appropriately, one obtains the master equations of the same form as those for the $(N-1)$-order comb, but with the renormalised hopping rates:

$$
\begin{array}{lll}
a' = ab/A & b' = b^2(1-rs)/B & c' = ds^2(1-de)/B \\
d' = ds^2(1-de)/B & e' = er^2/(1-rs-uv) & f' = f^2(1-rs)/B \\
g'\Delta' = f_g\Delta/A & h' = hs^2(1-de)/B & r' = r^3/(1-3rs) \\
s' = s^3/(1-3rs) & u' = ur^2/(1-rs-uv) & v' = vs^2/(1-rs-uv) \\
w' = wr^2/(1-3rs) & &
\end{array}
\tag{4}
$$

where $A = 1 - de - ac$ and $B = (1-de)(1-hw-rs) - 2bf(1-rs)$. Noting that g and Δ always appear together as the product $g\Delta$ for the solution for the first-passage probability, hereinafter we set $\Delta \equiv 1$ for simplicity. Repeating this procedure for N times, we get the master equations of the zeroth-order comb as (2) with a, c, d, e and g replaced by $a^{(N)}$, $c^{(N)}$, $d^{(N)}$, $e^{(N)}$ and $g^{(N)}$. So the solution for the first-passage probability for the Nth-order comb is

$$P_3^{(N)}(z) = \frac{c^{(N)}g^{(N)}}{1 - a^{(N)}c^{(N)} - d^{(N)}e^{(N)}}. \tag{5}$$

The qth moment of FPT is formally obtained from

$$\langle t^q \rangle_N = \left(z\frac{\mathrm{d}}{\mathrm{d}z} \right)^q P_3^{(N)}(z)|_{z=1}. \tag{6}$$

Computing the series expansion of $P_3^{(N)}(z)$ in powers of $\varepsilon = 1 - z$

$$P_3^{(N)}(z) = 1 + \sigma_1^{(N)}\varepsilon + \sigma_2^{(N)}\varepsilon^2 + \ldots \tag{7}$$

and applying (6), we identify:

$$\sigma_1^{(N)} = -\langle t \rangle_N \qquad \sigma_2^{(N)} = \tfrac{1}{2}(\langle t^2 \rangle_N - \langle t \rangle_N) \tag{8}$$

etc. To obtain the scaling properties of these moments, we first expand each of the hopping rates in powers of $\varepsilon = 1 - z$, e.g.

$$a^{(N)} = a_0^{(N)}(1 + a_1^{(N)}\varepsilon + a_2^{(N)}\varepsilon^2) + \mathrm{O}(\varepsilon^3) \tag{9}$$

and similarly for other hopping rates. With the use of recursion relation (4), we get the following relations after some algebra:

$$\sigma_1^{(N)} = a_1^{(N+1)} \qquad \sigma_2^{(N)} = a_2^{(n+1)}. \tag{10}$$

The recursion relations for a_1, a_2, b_1, b_2, etc are given by a direct calculation from (4), e.g.

$$
\begin{aligned}
a_1' &= \left(\frac{a_0}{g_0} + 1 \right)(a_1 + b_1) + \frac{d_0 e_0}{c_0 g_0}(d_1 + e_1) \\
a_2' &= \left(\frac{a_0}{g_0} + 1 \right)(a_2 + b_2 + a_1 b_1) + \frac{d_0 e_0}{c_0 g_0}(d_2 + e_2 + d_1 e_1) + \left(\frac{a_0}{g_0}(a_1 + b_1) + \frac{d_0 e_0}{c_0 g_0}(d_1 + e_1) \right) a_1'.
\end{aligned}
\tag{11}
$$

Comparing the master equation for $P_m(n)$ with those for the corresponding generating function $P_m(z)$, we obtain the initial value of the hopping rates:

$$
\begin{array}{lll}
a^{(0)} = b^{(0)} = p_2 z & c^{(0)} = d^{(0)} = v^{(0)} = z & f^{(0)} = g^{(0)} = p_1 z \\
e^{(0)} = w^{(0)} = p_3 z & h^{(0)} = s^{(0)} = \dfrac{(1+\delta_y)}{2} z & u^{(0)} = r^{(0)} = \dfrac{(1-\delta_y)}{2} z
\end{array}
\tag{12}
$$

with

$$p_1=\frac{2-\alpha(1-\delta_y)}{4}+\frac{\delta_z}{3} \qquad p_2=\frac{2-\alpha(1-\delta_y)}{4}-\frac{\delta_x}{3} \qquad p_3=\frac{(1-\delta_y)\alpha}{2}. \tag{13}$$

Here δ_x and δ_y are the biasing terms in the backbone and the tooth direction, respectively, and α is a parameter introduced to take into account the influence of the bias in the tooth direction on the hopping rates from the backbone sites to the teeth sites; thus it depends on δ_y as $\alpha=\alpha(\delta_y)$, with $\alpha(0)=\frac{2}{3}$.

With the use of (4)-(13), one can discuss the scaling of the first and second moments of FPT for various biases.

(i) For $\delta_x=\delta_y=0$, i.e. the diffusion without bias, a straightforward calculation shows that

$$a_0\equiv e_0\equiv g_0\equiv\tfrac{1}{3} \qquad r_0\equiv s_0\equiv u_0\equiv\tfrac{1}{2}v_0\equiv\tfrac{1}{2} \tag{14}$$

and the dominant contributions for d_0/c_0, e_1 and e_2 rescale as

$$\frac{d_0'}{c_0'}=\frac{2}{3}\frac{d_0}{c_0} \qquad e_1'=9e_1 \qquad e_2'\sim 81e_2. \tag{15}$$

With the dominant asymptotics $a_1\sim(d_0/c_0)e_1$ and $a_2\sim(d_0/c_0)e_2$ one has the scaling laws $\langle t\rangle_{N+1}=6\langle t\rangle_N$ and $\langle t^2\rangle_{N+1}\sim 54\langle t^2\rangle_N$, which have been obtained by Kahng *et al* in [4].

(ii) For $\delta_x=\delta>0$ and $\delta_y=0$, i.e. the bias field is introduced in the backbone direction to favour the diffusion, with the use of the recursion relations (4) and after some algebra, one has

$$a_0^{(N)}=\frac{1+\delta}{3}\left(\frac{1-\delta}{1+\delta}\right)^{2^N}. \tag{16}$$

Retaining only the asymptotically dominant contributions under rescaling leads to $a_1\sim(d_0/c_0)e_1$ and $a_2\sim(d_0/c_0)e_2$ with

$$\frac{d_0'}{c_0'}\sim\frac{1}{3}\frac{d_0}{c_0} \qquad e_1'=9e_1 \qquad e_2'\sim 81e_2. \tag{17}$$

Thus one has $\langle t\rangle_{N+1}\sim 3\langle t\rangle_N$ and $\langle t^2\rangle_{N+1}\sim 27\langle t^2\rangle_N$.

(iii) For $\delta_x>0$ and $\delta_y\neq 0$, from the initial value (12), (13) and the recursion relations for r, s, d and e, we have $r_0s_0\to 0$ and $d_0e_0\to 0$ upon rescaling. A tedious but straightforward calculation yields

$$\begin{aligned} a_1'&\sim a_1+b_1 \qquad & a_2'&\sim a_2+b_2+a_1b_1\\ b_1'&\sim 2b_1 \qquad & b_2'&\sim 4b_2 \end{aligned} \tag{18}$$

and then $\langle t\rangle_{N+1}\sim 2\langle t\rangle_N$, $\langle t^2\rangle_{N+1}\sim 4\langle t^2\rangle_N$.

(iv) For $\delta_x=0$, but $\delta_y\neq 0$, i.e. the bias is introduced in the tooth direction, similarly, with $d_0e_0\to 0$ and $r_0s_0\to 0$ under rescaling, one has

$$\begin{aligned} a_1'&\sim 2a_1+2b_1 \qquad & a_2'&\sim 2(a_2+b_2+a_1b_1)+2(a_1+b_1)^2\\ b_1'&\sim 4b_1 \qquad & b_2'&\sim 16b_2 \end{aligned} \tag{19}$$

which leads to $\langle t\rangle_{N+1}\sim 4\langle t\rangle_N$ and $\langle t^2\rangle\sim 16\langle t^2\rangle_N$.

(v) For $\delta_x = -\delta < 0$, in a similar way to the case (2), we have

$$a_0^{(N)} = p_1\left(\frac{p_2}{p_1}\right)^{2^N}. \tag{20}$$

Retaining only the asymptotically dominant terms leads to

$$a_1' \sim \left(\frac{a_0}{g_0}\right)a_1 \qquad a_2' \sim \left(\frac{a_0}{g_0}\right)^2 a_1^2. \tag{21}$$

Thus one gets the exponential growing

$$\langle t\rangle \sim \left(\frac{p_2}{p_1}\right)^{L} \qquad \langle t^2\rangle \sim \left(\frac{p_2}{p_1}\right)^{2L} \tag{22}$$

which is the same as on a straight segment [7].

For general R, the scaling exponent τ_q for the qth moment of FPT can be obtained as follows by a similar procedure†:

(i) for $\delta_x = \delta_y = 0$ [4]

$$\tau_q = \begin{cases} \ln 2R^{(2q-1)}/\ln 2 & R > 2 \\ 2q & R \leqslant 2 \end{cases} \tag{23}$$

(ii) for $\delta_x > 0$ and $\delta_y = 0$

$$\tau_q = \begin{cases} \ln R^{(2q-1)}/\ln 2 & R \neq 1 \\ q & R = 1 \end{cases} \tag{24}$$

(iii) for $\delta_x > 0$ and $\delta_y > 0$

$$\tau_q = q \tag{25}$$

(iv) for $\delta_x = 0$ and $\delta_y > 0$

$$\tau_q = 2q \tag{26}$$

(v) for $\delta_x < 0$:

$$\langle t^q\rangle_L \sim \left(\frac{p_2}{p_1}\right)^{qL}. \tag{27}$$

Thus we have found that the effect of bias depends only on its direction. In the anomalous regime ($R > 2$) [4], the introduction of the bias favouring diffusion in the tooth direction will cause a transition from anomalous to ordinary diffusion. This bias rounds off the dangling end effect of the hierarchical comb structures and eliminates the multifractal nature of diffusion. On the other hand, the effect of the bias favouring diffusion in the backbone direction depends on whether or not there is a bias field in the tooth direction. In the absence of bias in the tooth direction, the bias favouring diffusion in the backbone direction changes the scaling exponents for the FPT moments, from $\tau_q = 1 + (2q-1)\ln R/\ln 2$ to $\tau_q = (2q-1)\ln \mathrm{R}/\ln 2$. It does not eliminate the multifractal nature of diffusion. In contrast, when the hierarchical parameter $R = 2$, where the transition from ordinary to anomalous diffusion occurs and the multifractal behaviour disappears, the introduction of the favourable bias in the backbone direction

† Although we cannot provide a strict proof of these results for any q within the present context, it appears reasonable and physically clear to expect their validity (cf [4]).

will give rise to a new multifractality with $\tau_q = (2q-1)$. Finally, in the presence of bias not favouring the diffusion in the backbone direction, with $\delta_x < 0$, the moments of FPT grow exponentially as on a straight finite segment without any dangling end [7].

To conclude, we have investigated biased diffusion on the hierarchical comb structures using an exact renormalisation decimation technique and obtained the scaling exponents of the moments of FPT. On one hand, the bias favouring diffusion in the tooth direction and the bias not favouring diffusion in the backbone direction would round off the dangling end effect of the hierarchical comb and eliminate the multifractal nature of diffusion. On the other hand, the bias favouring diffusion in the backbone direction changes only the scaling exponents of the moments of FPT and does not eliminate the multifractal behaviour. In contrast, it may give rise to a new multifractality on the $R = 2$ hierarchical comb, where in the absence of bias, the scaling of the moments of FPT could be characterised by a single exponent.

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